Teacher notes
Topic C

Energy conservation and resonance

This note addresses three of the linking questions to Topic C4. The first is:

How can resonance be explained in terms of conservation of energy?

It is a good question but any discussion of it requires material way above the syllabus so it is doubtful if it will ever be discussed and will certainly never be examined except perhaps at a superficial level. For what it is worth, here is an analysis of the situation.

We consider the typical oscillator described by the equation

$$\frac{d^2x}{dt^2}+\omega_0^2 x=\frac{F}{m}\cos\omega t \qquad \text{(Eq. 1)}$$

where the natural frequency of the oscillator is ω_0 and an external periodic force $F\cos\omega t$ acts on the oscillator. We ignore damping for now.

Multiply by $\frac{dx}{dt}$ to get

$$m\frac{d^2x}{dt^2}\frac{dx}{dt}+m\omega_0^2 x\frac{dx}{dt}=F\frac{dx}{dt}\cos\omega t$$
$$\frac{d}{dt}\left(\frac{1}{2}m(\frac{dx}{dt})^2+\frac{1}{2}m\omega_0^2x^2\right)=F\frac{dx}{dt}\cos\omega t$$

The quantity in big brackets, $\frac{1}{2}m(\frac{dx}{dt})^2+\frac{1}{2}m\omega_0^2x^2$, is the total energy *E* of the oscillator (kinetic plus elastic potential).

Integrating with respect to time we find

$$\Delta E=\int_0^T F\frac{dx}{dt}\cos\omega t\ dt=\int_0^X F\cos\omega t\ dx$$

The last term on the right is the work done by the external force. In other words, the change in total energy of the oscillator in time *T* is equal to the work done by the external periodic force over a distance *X* (the distance the external force moves its point of application in time *T*).

The solution to Eq. 1 for $\omega\neq\omega_0$ is

$$x = \frac{F}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

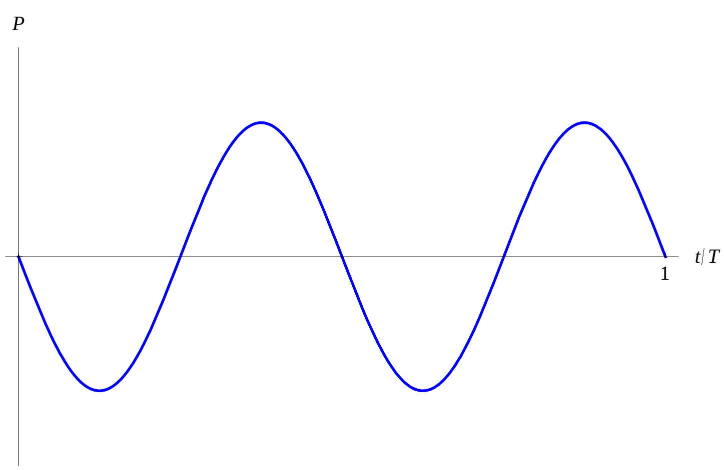
We see that the oscillator oscillates with the frequency of the external force as expected. The amplitude is constant. Now apply:

$$\Delta E = \int_0^T F \frac{dx}{dt} \cos \omega t \, dt$$

using $\frac{dx}{dt} = -\frac{F\omega}{m(\omega_0^2 - \omega^2)} \sin \omega t$ to get

$$\begin{aligned}\Delta E &= -\int_0^T \frac{F^2 \omega}{m(\omega_0^2 - \omega^2)} \sin(\omega t) \cos(\omega t) \, dt \\ &= -\frac{1}{2} \frac{F^2 \omega}{m(\omega_0^2 - \omega^2)} \int_0^T \sin(2\omega t) \, dt\end{aligned}$$

The power supplied varies according to the graph below, shown for **one** full period.



We see that as much energy goes in (positive power) as goes out (negative power). So, on average the total energy does not change. The external force provides energy but also takes energy away.

Continuing the derivation for the change in energy:

$$\begin{aligned}\Delta E &= +\frac{1}{2} \frac{F^2 \omega}{m(\omega_0^2 - \omega^2)} \left. \frac{\cos(2\omega t)}{2\omega} \right|_0^T \\ &= \frac{1}{4} \frac{F^2}{m(\omega_0^2 - \omega^2)} (\cos 2\omega T - 1)\end{aligned}$$

Explicitly:

The change in total energy in the first quarter of a period is

$$\Delta E = \frac{1}{4}\frac{F^2}{m(\omega_0^2-\omega^2)}(\cos 2\omega\frac{2\pi}{4\omega}-1) = \frac{1}{4}\frac{F^2}{m(\omega_0^2-\omega^2)}(-1-1) = -\frac{1}{2}\frac{F^2}{m(\omega_0^2-\omega^2)}$$

In the second quarter it is

$$\Delta E = \frac{1}{4}\frac{F^2}{m(\omega_0^2-\omega^2)}(\cos 2\omega\times 2\times\frac{2\pi}{4\omega}-\cos 2\omega\times\frac{2\pi}{4\omega}) = \frac{1}{4}\frac{F^2}{m(\omega_0^2-\omega^2)}(1+1) = +\frac{1}{2}\frac{F^2}{m(\omega_0^2-\omega^2)}$$

In the third it is

$$\Delta E = \frac{1}{4}\frac{F^2}{m(\omega_0^2-\omega^2)}(\cos 2\omega\times 3\times\frac{2\pi}{4\omega}-\cos 2\omega\times 2\times\frac{2\pi}{4\omega}) = \frac{1}{4}\frac{F^2}{m(\omega_0^2-\omega^2)}(-1-1) = -\frac{1}{2}\frac{F^2}{m(\omega_0^2-\omega^2)}$$

In the fourth

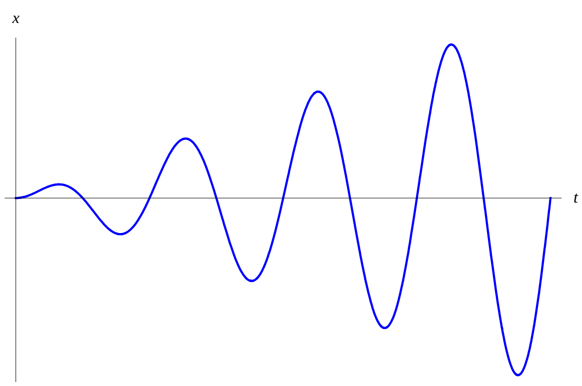
$$\Delta E = \frac{1}{4}\frac{F^2}{m(\omega_0^2-\omega^2)}(\cos 2\omega\times 4\times\frac{2\pi}{4\omega}-\cos 2\omega\times 3\times\frac{2\pi}{4\omega}) = \frac{1}{4}\frac{F^2}{m(\omega_0^2-\omega^2)}(1+1) = \frac{1}{2}\frac{F^2}{m(\omega_0^2-\omega^2)}$$

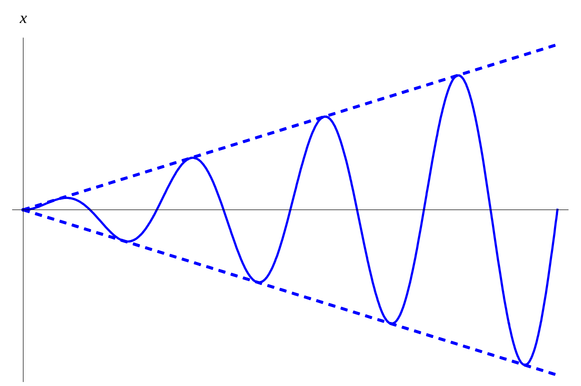
This means that during one period the total energy does not change. These results agree with the graph for power above.

The case of resonance

We now consider the solution to Eq. 1 for $\omega = \omega_0$.

The solution is $x = \frac{F}{2m\omega_0}t\sin\omega_0 t$. This is graphed below. The amplitude is increasing as expected.





We see that the amplitude increases proportionally with time (graph to the right). So, we expect the energy to increase proportionally to time squared. We see this explicitly as:

Since $\frac{dx}{dt} = \frac{F}{2m\omega_0}(\sin\omega_0 t + \omega_0 t\cos\omega_0 t)$ repeating the steps above for the energy change we get

$$\Delta E = \int_{T_1}^{T_2} \frac{F^2}{2m\omega_0}(\sin\omega_0 t + \omega_0 t\cos\omega_0 t)\cos(\omega_0 t)\,dt$$
$$= \frac{F^2}{2m\omega_0}\int_{T_1}^{T_2}(\sin\omega_0 t\cos\omega_0 t + \omega_0 t\cos^2\omega_0 t)\,dt$$

The first term is like that in the non-resonant case and so on average does not change the total energy. But the second does.

In fact:

$$\Delta E = \frac{F^2}{2m}\int_{T_1}^{T_2}(t\cos^2\omega_0 t)\,dt = \frac{F^2}{2m}\left(\frac{t^2}{4} + \frac{\cos 2\omega_0 t}{8\omega_0^2} + \frac{t\sin 2\omega_0 t}{4\omega_0}\right)$$

The dominant term is the first, so that energy increases at a rate proportional to t^2.

So, the linking question is answered by saying that at resonance the amplitude increases, so the energy increases, and the extra energy is provided by the work done by the external force.

This brings us to another linking question in C4:

How does the amplitude of vibration at resonance depend on the dissipation of energy in the driven system?

Now we have damping. The previous analysis can be repeated but it is a bit more complicated. The answer is that now the amplitude is constant and so the energy is constant as well. This is because the rate at which energy is being provided to the system is equal to the rate at which energy is dissipated from the system.

Finally, we can address another linking question from C4:

What is the relationship between resonance and simple harmonic motion?

As we saw the motion of the system at resonance (without damping) occurs at the frequency of the driving external force, but the amplitude is not constant. The motion is periodic, like in SHM, but the amplitude is increasing unlike SHM.

Ass mentioned at the beginning, these are very good linking questions but impossible to discuss meaningfully because we do not have the mathematical machinery to address them in this course.